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LETTER TO THE EDITOR

A general formula for solition form factors in the quantum sine-Gordon model

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Abstract. A general formula for soliton form factors of several local fields in the quantum sine-Gordon model is presented.

This letter is a continuation of an earlier preprint (Smirnov 1986). Consider the quantum sine-Gordon model with the Hamiltonian

$$H = \int \left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \frac{m^2}{8\gamma}[1 - \cos 2(2\gamma)^{1/2}u]\right) \mathrm{d}x.$$

It is already known (Faddeev *et al* 1979) that the spectrum of the model involves solitons and breathers, the latter being the bound states of the former. Solitons possess an internal degree of freedom (soliton, antisoliton). Here a formula will be given for the matrix element (form factor) of the quantum Just function (Smirnov 1984) taken between the vacuum and a normalised 'in' state containing $2n$ solitons with rapidities $\beta_1 < \beta_2 < \ldots < \beta_{2n}$. A different order of rapidities can also be considered due to the symmetry. The form factor in question $f(x, t|\sigma|\beta_1, \ldots, \beta_{2n})$ is a covector in the space $H_n^* = h_1^* \otimes \ldots \otimes h_{2n}^* (h_i = \mathbb{C}^2)$ of internal states of solitons. It depends on x, t, $\beta_1, \ldots, \beta_{2n}$ and an additional parameter σ. Denote the basis in h_i^* by $e_\varepsilon^{(i)}$ ($\varepsilon = \pm 1$). In the natural basis of the tensor product $f(x, t|\sigma|\beta_1, \ldots, \beta_{2n})$ has C_{2n}^n non-zero components, namely the components with $\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{2n} = 0$, i.e. the number of solitons is equal to the number of antisolitons. The form factor $f(x, t|\sigma|\beta_1, \ldots, \beta_{2n})$ is a periodic function of σ with period $(2\pi - 2\gamma)\mathrm{i}$. Form factors of local fields can be obtained as the coefficients in the expansion of $f(x, t|\sigma|\beta_1, \ldots, \beta_{2n})$ as a function of $\exp(p\sigma)$ ($p = \pi/(\pi - \gamma)$) into the series in the neighbourhood of the points $\exp(p\sigma) = 0$, $\exp(p\sigma) = \infty$. In particular, the form factor of $\exp(-\mathrm{i}(2\gamma)^{1/2}u(x, t))$ is equal to $f(x, t|-\infty|\beta_1, \ldots, \beta_{2n})$ and the form factors of $u_x - u_t$, $\int_{-\infty}^{x+t} (\cos 2(2\gamma)^{1/2}u - 1)\,\mathrm{d}\xi_+$ are equal to $\tilde{f}(x, t|\beta_1, \ldots, \beta_{2n}) \times (1 \pm \Pi_{j=1}^{2n} \otimes \sigma_i^1)$ respectively, where σ_i^1 is the Pauli matrix σ^1 acting in h_i and

$$\tilde{f}(x, t|\beta_1, \ldots, \beta_{2n}) = \lim_{\sigma \to \infty} (f(x, t|\sigma|\beta_1, \ldots, \beta_{2n}) - \delta_{n,0}) \exp\left(p\sigma - \tfrac{1}{2}p \sum_{j=1}^{2n} \beta_j\right).$$

Now we pass to an explicit description of $f(x, t|\sigma|\beta_1, \ldots, \beta_{2n})$. First, let us extract from $f(x, t|\sigma|\beta_1, \ldots, \beta_{2n})$ some simple factors

$$f(x, t|\sigma|\beta_1, \ldots, \beta_{2n}) = \exp\left(-\mathrm{i}\sum_{j=1}^{2n} P_\mu(\beta_j)x_\mu\right)\prod_{i<j}^{2n}\zeta(\beta_{ij}) \times \prod_{j=1}^{2n}[\tanh\tfrac{1}{2}p(\sigma-\beta_j-\tfrac{1}{2}\mathrm{i}\pi)-1]h(\sigma|\beta_1, \ldots, \beta_{2n})$$

where

$$\beta_{ij} = \beta_i - \beta_j$$

$$\zeta(\beta) = \sinh\tfrac{1}{2}p\beta\,\exp\left(\int_0^\infty \frac{\sin^2\frac{1}{2}\kappa(\beta+\mathrm{i}\pi-\mathrm{i}\gamma)\sinh(\frac{1}{2}\pi-\gamma)\kappa}{\kappa\sinh\frac{1}{2}\gamma\kappa\,\sinh(\pi-\gamma)\kappa\,\cosh\frac{1}{2}(\pi-\gamma)\kappa}\,\mathrm{d}\kappa\right)$$

$$p(\beta) = m_\mathrm{s}(\cosh p\beta, \sinh p\beta)$$

m_s being the soliton mass. To describe $h(\sigma|\beta_1, \ldots, \beta_{2n})$ some notation is needed. We denote by $\hat{S}_{i,j}(\beta)$ the operator which acts non-trivially only in the spaces h_i and h_j and is given by

$$\hat{S}_{i,j}(\beta) = \tfrac{1}{2}\sinh^{-1}\frac{\pi}{\gamma}(\beta-\mathrm{i}\pi)\left(I_i\otimes I_j\left(\sinh\frac{\pi}{\gamma}(\beta-\mathrm{i}\pi)+\sinh\frac{\pi}{\gamma}\beta\right) + \sigma_i^3\otimes\sigma_j^3\left(\sinh\frac{\pi}{\gamma}(\beta-\mathrm{i}\pi)-\sinh\frac{\pi}{\gamma}\beta\right) - \sinh\frac{\pi^2\mathrm{i}}{\gamma}(\sigma_i^1\otimes\sigma_j^1+\sigma_i^2\otimes\sigma_j^2)\right)$$

where I_i, σ_i^1, σ_i^2, σ_i^3 are the unit matrix and Pauli matrices acting in the space h_i. Let us introduce an auxiliary space $h_0 \simeq \mathbb{C}^2$ and operators

$$\begin{pmatrix} A(\tau) & B(\tau) \\ C(\tau) & D(\tau)\end{pmatrix} = \hat{S}_{0,1}(\tau-\beta_1)\hat{S}_{0,2}(\tau-\beta_2)\ldots\hat{S}_{0,2n}(\tau-\beta_{2n}).$$

Here $\hat{S}_{0i}$ are multiplied as matrices in the space h_0 and the LHS is divided into blocks as a matrix in h_0. This object is common in the framework of the quantum inverse transform method (Faddeev 1980). Let $F(a_1, \ldots, a_n|b_1, \ldots, b_n)$ be a polynomial invariant with respect to the independent permutations of $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$. We define a covector

$$\langle F\rangle(\beta_1, \ldots, \beta_{2n}) = e_{-1}^{(1)}\otimes\ldots\otimes e_{-1}^{(2n)}\sum_{\{1,\ldots,2n\}=\{\kappa_1,\ldots,\kappa_n\}\cup\{l_1,\ldots,l_n\}} B(\beta_{\kappa_1})\ldots B(\beta_{\kappa_n}) \times\prod_{i,j=1}^{n}\sinh^{-1}\frac{\pi}{\gamma}(\beta_{\kappa_i}-\beta_{\kappa_j}) \times\exp\left(\frac{\pi}{2\gamma}\sum_{j=1}^{n}(\beta_{l_j}-\beta_{\kappa_j}-\mathrm{i}\pi)\right) \times F\left(\exp\left(\frac{2\pi}{\gamma}\beta_{\kappa_1}\right), \ldots, \exp\left(\frac{2\pi}{\gamma}\beta_{\kappa_n}\right)\middle|\exp\left(\frac{2\pi}{\gamma}\beta_{l_1}\right), \ldots, \exp\left(\frac{2\pi}{\gamma}\beta_{l_n}\right)\right).$$

An operation assigning $\langle F\rangle$ to F will be of importance later.

Now we introduce another useful operation. Denote by $\varphi(\beta)$ the function

$$\varphi(\beta)=\exp\left(\int_0^\infty \frac{2\sin^2\frac{1}{2}(\sigma+\frac{1}{2}\mathrm{i}\pi-\frac{1}{2}\mathrm{i}\gamma)\kappa\,\sinh(\frac{1}{2}\pi-\gamma)\kappa}{\kappa\sinh(\pi-\gamma)\kappa\,\sinh\frac{1}{2}\gamma\kappa}\,\mathrm{d}\kappa\right)$$

with the following asymptotics:

$$\varphi(\beta)=\mathrm{O}(\exp\tfrac{1}{2}(\pi/\gamma-p)|\beta|)\qquad |\beta|\sim\infty.$$

Consider an antisymmetric polynomial $P(y_1,\ldots,y_{n-1})$ which has a degree at most $2n-2$ in each of its arguments. Let $\phi(P)$ be the following integral transformation:

$$\begin{aligned}\phi(P)=&⨎_{\Gamma_2}\mathrm{d}\alpha_1\ldots⨎_{\Gamma_2}\mathrm{d}\alpha_{n-1}\prod_{i=1}^{n-1}\prod_{j=1}^{2n}\varphi(\alpha_i-\beta_j)\left[\varepsilon\exp\left(\frac{2\pi}{\gamma}\alpha_i\right)-\varepsilon^{-1}\exp\left(\frac{2\pi}{\gamma}\beta_j\right)\right]^{-1}\\&\times\prod_{i=1}^{n-1}(1-\exp\{p(\sigma-\alpha_i-\tfrac{1}{2}\mathrm{i}\pi)\})\exp\left(\frac{\pi}{\gamma}(n+2)\sum_{i=1}^{n-1}\alpha_i\right)\\&\times\prod_{i<j}^{n-1}\sinh p\alpha_{ij}P\left(\exp\frac{2\pi}{\gamma}\alpha_1,\ldots,\exp\frac{2\pi}{\gamma}\alpha_{n-1}\right)\end{aligned}$$

where $\varepsilon=\exp(\pi^2\mathrm{i}/\gamma)$, $\Gamma_2=\mathbb{R}-\mathrm{i}\pi+\mathrm{i}\gamma-\mathrm{i}0$ and the double crossed integral means the following regularisation:

$$\begin{aligned}&⨎_{\Gamma_2}\prod_{j=1}^{2n}\varphi(\alpha-\beta_j)\left[\varepsilon\exp\left(\frac{2\pi}{\gamma}\alpha\right)-\varepsilon^{-1}\exp\left(\frac{2\pi}{\gamma}\beta_j\right)\right]^{-1}\\&\qquad\times\exp\left[\left(p\kappa+\frac{\pi}{\gamma}(n+2)\right)\alpha\right]P\left[\exp\left(\frac{2\pi}{\gamma}\alpha\right)\right]\mathrm{d}\alpha\\&\quad=\int_{\Gamma_2}\prod_{j=1}^{2n}\varphi(\alpha-\beta_j)\left[\varepsilon\exp\left(\frac{2\pi}{\gamma}\alpha\right)-\varepsilon^{-1}\exp\left(\frac{2\pi}{\gamma}\beta_j\right)\right]^{-1}\\&\qquad\times\left[\exp\left(\frac{2\pi}{\gamma}\alpha\right)-\exp\left(\frac{2\pi}{\gamma}\beta_j\right)\right]^{-1}\exp\left[\left(p\kappa+\frac{\pi}{\gamma}(n+2)\right)\alpha\right]\\&\qquad\times P_1\left[\exp\left(\frac{2\pi}{\gamma}\alpha\right)\right]\mathrm{d}\alpha+\int_{\Gamma_1}\prod_{j=1}^{2n}\varphi(\alpha-\beta_j)\left[\exp\left(\frac{2\pi}{\gamma}\alpha\right)-\exp\left(\frac{2\pi}{\gamma}\beta_j\right)\right]^{-1}\\&\qquad\times P_2\left[\exp\left(\frac{2\pi}{\gamma}\alpha\right)\right]\exp\left[\left(p\kappa+\frac{\pi}{\gamma}(n+2)\right)\alpha\right]\mathrm{d}\alpha\end{aligned}\qquad(1)$$

where $|\kappa|\leqslant n-1$, $\deg P(y)=2n-2$, Γ_1 is a contour which encircles all the poles of the integrand in the strip $-\pi+\gamma-0\leqslant\mathrm{Im}\,\alpha\leqslant\pi-\gamma-0$ and the polynomials P_1, P_2 are defined by the relation

$$\begin{aligned}&\prod_{j=1}^{2n}\left[y-\exp\left(\frac{2\pi}{\gamma}\beta_j\right)\right]P(y)\\&\quad=P_1(y)+\prod_{j=1}^{2n}\left[\varepsilon y-\varepsilon^{-1}\exp\left(\frac{2\pi}{\gamma}\beta_j\right)\right]P_2(y)\\&\qquad-\varepsilon^{-2(n-2)}\prod_{j=1}^{2n}\left[y-\exp\left(\frac{2\pi}{\gamma}\beta_j\right)\right]P_2(y\varepsilon^4)\qquad \deg P_1(y)\leqslant 3n-1.\end{aligned}$$

One can show that it is always possible to find such polynomials. The definition of

P_1 and P_2 does not secure their uniqueness. However the last circumstance does not affect the uniqueness of the RHS in (1) which follows from the identity

$$\varphi(\alpha+2\pi i-2\gamma i)=\frac{\sinh(\pi/\gamma)(\alpha+\pi i)}{\sinh(\pi/\gamma)\alpha}\varphi(\alpha).$$

Now we are in a position to write down the formula for $h(\sigma|\beta_1,\ldots,\beta_{2n})$:

$$h(\sigma|\beta_1,\ldots,\beta_{2n})=\phi(\langle\Delta_n\rangle)$$

where

$$\Delta_n(a_1,\ldots,a_n|b_1,\ldots,b_n|y_1,\ldots,y_{n-1})=\det\|A_{ij}^{(n)}\|.$$

$\|A_{ij}^{(n)}\|$ is a $(n-1)\times(n-1)$ matrix with the following matrix elements:

$$A_{ij}^{(n)}=\prod_{l=1}^{n}(y_j-a_l\varepsilon^{-2})\left(\sum_{\kappa=0}^{i-1}(1-\varepsilon^{2(i-\kappa)})(-1)^{\kappa}y_j^{i-\kappa-1}\varepsilon^{-2\kappa}\sigma_\kappa(b_1,\ldots,b_n)\right)$$
$$+\varepsilon^{2i}\prod_{l=1}^{n}(y_j-\varepsilon^{-4}b_l)\left(\sum_{\kappa=0}^{i-1}(1-\varepsilon^{2(i-\kappa)})(-1)^{\kappa}y_j^{i-\kappa-1}\varepsilon^{-2\kappa}\sigma_\kappa(a_1,\ldots,a_n)\right)$$

where σ_κ is an elementary symmetric polynomial of degree κ.

Thus we have constructed $h(\sigma|\beta_1,\ldots,\beta_{2n})$, and consequently $f(x,t|\sigma|\beta_1,\ldots,\beta_{2n})$. It can be shown that $f(x,t|\sigma|\beta_1,\ldots,\beta_{2n})$ satisfies the relations

$$f(x,t|\sigma|\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots,\beta_{2n})S_{i,i+1}(\beta_{i,i+1})$$
$$=-f(x,t|\sigma|\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots,\beta_{2n})P_{i,i+1}$$
$$f(x,t|\sigma|\beta_1,\ldots,\beta_{2n}+2\pi i-2\gamma i)$$
$$=f(x,t|\sigma|\beta_1,\ldots,\beta_{2n})S_{2n-1,2n}(\beta_{2n-1}-\beta_{2n})\ldots S_{1,2n}(\beta_1-\beta_{2n})$$

where $P_{i,i+1}$ permutes the spaces h_i and h_{i+1} and $S_{i,i+1}(\beta)$ is the scattering matrix (Zamolodchikov 1977, Korepin 1979)

$$S_{i,i+1}(\beta)=S_0(\beta)\hat{S}_{i,i+1}(\beta)$$
$$S_0(\beta)=\exp\left(-i\int_0^\infty\frac{\sin\kappa\sigma\sinh(\frac{1}{2}\pi-\gamma)\kappa}{\kappa\cosh\frac{1}{2}(\pi-\gamma)\kappa\sinh\frac{1}{2}\gamma\kappa}\,d\kappa\right).$$

The necessity for $f(x,t|\sigma|\beta_1,\ldots,\beta_{2n})$ to satisfy these relations follows from results obtained by Smirnov (1986). The form factor $f(x,t|\sigma|\beta_1,\ldots,\beta_{2n})$ also satisfies some recursive relations (Smirnov 1986), which we do not write down because of their bulky character. A detailed proof of these statements will be given in a further, longer, publication where we will also give the formulae for the soliton and breather form factors. The latter can be calculated as residues of the soliton ones.

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